MODEL SOLUTIONS

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Candidate surname	Other names
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Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Monday 26 June 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/4A**

Further Mathematics

Advanced

PAPER 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistics Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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1.

$$\mathbf{A} = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$$

where a is a constant.

(a) Determine, in expanded form in terms of a, the characteristic equation for $\mathbf{A}$.

(2)

(b) Hence use the Cayley-Hamilton theorem to determine values of a and b such that

$$\mathbf{A}^3 = \mathbf{A} + b\mathbf{I}$$

where $\mathbf{I}$ is the 2×2 identity matrix.

(4)

a) $\det(A - \lambda I) = 0$

(1) $\begin{vmatrix} -1-\lambda & a \\ 3 & 8-\lambda \end{vmatrix} = (-1-\lambda)(8-\lambda) - 3a$

$= -8 - 8\lambda + \lambda + \lambda^2 - 3a$

$= \lambda^2 - 7\lambda + (-3a - 8) = 0$ (1)

b) Replace λ with A and multiply units by I.

$A^2 - 7A - (8+3a)I = 0$ (1)

Multiply everything by A, so we have an A^3 term.

$\Rightarrow A^3 - 7A^2 = (8+3a)A$ (1)

If we rearrange (1) for A^2, we can sub this in.

$\Rightarrow A^3 - 7(7A + (8+3a)I) = (8+3a)A$ (1)

$\Rightarrow A^3 - 49A - (56+21a)I = (8+3a)A$

$\Rightarrow A^3 = (57 + 3a)A + (56 + 21a)I$

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Question 1 continued

$\Rightarrow \quad 57 + 3a = 1$

$\Rightarrow \quad a = -\dfrac{56}{3}$ (1)

$\Rightarrow \quad b = 56 + 21\left(-\dfrac{56}{3}\right)$

$\Rightarrow \quad b = -366$ (1)

(Total for Question 1 is 6 marks)

2. A complex number z is represented by the point P in the complex plane.

Given that z satisfies

$$|z - 6| = 2|z + 3\mathrm{i}|$$

(a) show that the locus of P passes through the origin and the points -4 and $-8\mathrm{i}$ **(2)**

(b) Sketch on an Argand diagram the locus of P as z varies. **(2)**

(c) On your sketch, shade the region which satisfies both

$$|z - 6| \geqslant 2|z + 3\mathrm{i}| \quad \text{and} \quad |z| \leqslant 4$$

(2)

a) $|z - 6| = 2|z + 3iy|$

Let $z = x + iy$

$\Rightarrow |x + iy - 6| = 2|x + iy + 3iy|$

$\Rightarrow |x - 6 + iy| = 2|x + i(y+3)|$ $\quad |a+ib| = \sqrt{a^2+b^2}$

$\Rightarrow \sqrt{(x-6)^2 + y^2} = 2\sqrt{x^2 + (y+3)^2}$

$\Rightarrow (x-6)^2 + y^2 = 4(x^2 + (y+3)^2)$

$\Rightarrow x^2 - 12x + 36 + y^2 = 4x^2 + 4y^2 + 24y + 36$

$\Rightarrow 0 = 3x^2 + 12x + 3y^2 + 24y$

$\Rightarrow 0 = x^2 + 4x + y^2 + 8y$

$\Rightarrow 0 = (x+2)^2 - 4 + (y+4)^2 - 16$

$\Rightarrow (x+2)^2 + (y+4)^2 = 20$ 1

Sub in $(x, y) = (0, 0)$

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Question 2 continued

$(0+2)^2+(0+4)^2=4+16=20,$

Hence, passes through the origin.

Sub in $(x,y)=(-4,0)$

$(-4+2)^2+(0+4)^2=4+16=20$

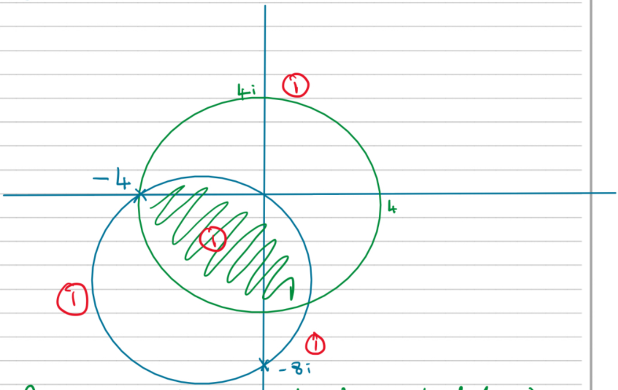
Hence, passes through $(-4,0)$.

Sub in $(x,y)=(0,-8)$

$(0+2)^2+(-8+4)^2=4+16=20$

Hence, passes through $(0,-8)$ 1

b) • Blue circle.



c) Draw a circle of radius 4, centred (0,0)

3. In a model for the number of subscribers to a new social media channel it is assumed that

- each week 20% of the subscribers at the start of the week cancel their subscriptions
- between the start and end of week n the channel gains $20n$ new subscribers

Given that at the end of week 1 there were 25 subscribers,

(a) explain why the number of subscribers at the end of week n, U_n, is modelled by the recurrence relation

$$U_1 = 25 \qquad U_{n+1} = 0.8U_n + 20(n+1) \qquad n = 1, 2, 3, \ldots$$

(2)

(b) Prove by induction that for $n \geqslant 1$

$$U_n = 325\left(\frac{4}{5}\right)^{n-1} + 100n - 400$$

(5)

Given that 6 months after starting the channel there were approximately 1800 subscribers,

(c) evaluate the model in the light of this information.

(2)

a) • $U_1 = 25$, because there are 25 subscribers at the end of week 1.

• 20% of the subscribers leave, meaning that 80% remain, so $0.8U_n$.

• At the end of week $n+1$, there are $20(n+1)$ subscribers added to those from week n. 1

All of the three points above put together is conclusion. Hence,

$U_{n+1} = 0.8U_n + 20(n+1)$, $U_1 = 25$ 1

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Question 3 continued

b) Base Case, let $n=1$.

$U_1 = 325\left(\frac{4}{5}\right)^{1-1} + 100(1) - 400$

$= 325 + 100 - 400 = 25$

So, true for $n=1$ 1

Assume true for $n=k$. Then

$U_k = 325\left(\frac{4}{5}\right)^{k-1} + 100k - 400$

Inductive Step, let $n=k+1$. Then, using the formula in part a,

$U_{k+1} = \frac{4}{5}U_k + 20(k+1)$

$= \frac{4}{5}\left(325\left(\frac{4}{5}\right)^{k-1} + 100k - 400\right) + 20k + 20$ 1

$= 325\left(\frac{4}{5}\right)^{k} + 80k - 320 + 20k + 20$

$= 325\left(\frac{4}{5}\right)^{k} + 100k - 300$ 1

$= 325\left(\frac{4}{5}\right)^{(k+1)-1} + 100(k+1) - 400$ 1

Hence, if the result is true for $n=k$, then it has been proven to be true for $n=k+1$. As it is true for $n=1$, it must be true for all positive integers n.

1

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Question 3 continued

c) n is measured in weeks, 6 months is approximately 24 weeks.

$U_{26} = 325(0.8)^{25} + 100(26) - 400$

$= 2201.227$ ①

The model is not great, as this is an overestimate. ①

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4. (a) Use the Euclidean algorithm to show that the highest common factor of 168 and 66 is 6

(2)

(b) Use back substitution to determine integers a and b such that

$$168a + 66b = 6$$

(3)

(c) Explain why there are no integer solutions to the equation

$$168x + 66y = 10$$

(1)

(d) Solve the congruence equation

$$11v \equiv 8 \pmod{28}$$

(3)

a) $168 - 2(66) = 36$

$66 - 1(36) = 30$

$36 - 1(30) = 6$

$30 - 5(6) = 0$ 1

The last non-zero remainder is 6, so 6 is the HCF. 1

b) We will sub in values we had before.

$36 - 30 = 6$ 1

$\Rightarrow 36 - (66 - 36) = 6$ 1

$\Rightarrow 2(36) - 66 = 6$

$\Rightarrow 2(168 - 2(66)) - 66 = 6$

$\Rightarrow 2(168) - 5(66) = 6$

$\Rightarrow a = 2,\ b = -5$ 1

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Question 4 continued

c) As the HCF of 168 and 66 is 6, the RHS has to be a multiple of 6 in order to have integer solutions. 10 is not a multiple of 6, hence there are no integer solutions. 1

d) $28 - 2(11) = 6$

$11 - 6 = 5$

$6 - 5 = 1$

So $HCF(11, 28) = 1$

We complete another back substitution.

$6 - (11 - 6) = 1$

$\Rightarrow 2(6) - 11 = 1$

$\Rightarrow 2(28 - 2(11)) - 11 = 1$

$\Rightarrow 2(28) - 5(11) = 1$ 1

$\Rightarrow -5(11) = 1 - 2(28)$

$\Rightarrow -5(11) \equiv 1 \pmod{28}$ 1

Multiply the equation in the question by −5.

$v \equiv -40 \pmod{28}$

$\Rightarrow v \equiv 16 \pmod{28}$ 1

5. (i) A security code is made up of 4 numerical digits followed by 3 **distinct** uppercase letters.

Given that the digits must be from the set {1, 2, 3, 4, 5} and the letters from the set {A, B, C, D}

(a) determine the total number of possible codes using this system.

To enable more codes to be generated, the system is adapted so that the 3 letters can appear anywhere in the code **but** no letter can be next to another letter.

(b) Determine the increase in the number of codes using this adapted system.

(4)

(ii) A combination lock code consists of four **distinct** digits that can be read as a positive integer, $N = abcd$, satisfying

- all the digits are odd
- N is divisible by 9
- the digits appear in either ascending or descending order
- $N \equiv e \pmod{ab}$ where ab is read as a two-digit number and e is the odd digit that is not used in the code

(a) Use the first two properties to determine the four digits used in the code.

(b) Hence determine the code on the lock.

(4)

The letters are distinct

i)a) $5^4 \times 4 \times 3 \times 2 = 15000$ 1

Any 5 numbers, not distinct

b) ↑ a ↑ b ↑ c ↑ d ↑

Let $a, b, c, d \in \{1, 2, 3, 4, 5\}$. 3 letters A, B, C, D can go in the 5 gaps. So we have

$\left(\binom{5}{3} \times 4 \times 3 \times 2\right) \times 5^4 - 15000$ 1

because we are finding the increase.

$= 150000 - 15000$

$= 135000$ 1

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Question 5 continued

ii)a) by (1), the only numbers can be 1, 3, 5, 7, 9

by (2), $1+3+5+7 \neq 9k$ (1)
$1+3+5+9 = 9k$
$1+3+7+9 \neq 9k$
$1+5+7+9 \neq 9k$
$3+5+7+9 \neq 9k$

So, there cannot be a 7.

Hence, the numbers are 1, 3, 5, 9. (1)

ii) by (3), the only possible combinations are

1359 or 9531

We just try 1359 and if it works for (4), then we know it is the code, otherwise 9531 is the code.

$1359 \equiv 7 \bmod (13)$ (1)

$1359 \div 13 = 104 \frac{7}{13}$

Hence, 1359 is the code. (1)

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6. Determine a closed form for the recurrence relation

$$u_0 = 1 \qquad u_1 = 4$$

$$u_{n+2} = 2u_{n+1} - \frac{4}{3}u_n + n \qquad n \geqslant 0$$

(7)

$U_{n+2} - 2U_{n+1} + \frac{4}{3}U_n = n \quad (1)$

$\Rightarrow \lambda^2 - 2\lambda + \frac{4}{3} = 0$ 1 Auxillary Equation

$\Rightarrow (\lambda - 1)^2 + \frac{1}{3} = 0$

$\Rightarrow \lambda = 1 \pm \frac{1}{\sqrt{3}} i$ 1

$\Rightarrow U_n = A\left(1 + \frac{1}{\sqrt{3}} i\right)^n + B\left(1 - \frac{1}{\sqrt{3}} i\right)^n$ 1 CF

$U_n = kn + c$

$U_{n+1} = k(n+1) + c$ 1

$U_{n+2} = k(n+2) + c$

Sub these into (1)

$k(n+2) + c - 2(k(n+1) + c) + \frac{4}{3}(kn + c) = n$

$\Rightarrow kn + 2k + c - 2kn - 2k - 2c + \frac{4}{3}kn + \frac{4}{3}c = n$

Now we compare coefficients.

'n' : $k - 2k + \frac{4}{3}k = 1 \Rightarrow k = 3$ 1

'1' : $2k + c - 2k - 2c + \frac{4}{3}c = 0 \Rightarrow c = 0$

$U_n = A\left(1 + \frac{1}{\sqrt{3}} i\right)^n + B\left(1 - \frac{1}{\sqrt{3}} i\right)^n + 3n$

Sub in the initial conditions.

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Question 6 continued

$$1 = A\left(1+\frac{1}{\sqrt{3}}i\right)^1 + B\left(1-\frac{1}{\sqrt{3}}i\right)^1$$

$$\Rightarrow 1 = A+B$$

$$4 = A\left(1+\frac{1}{\sqrt{3}}i\right)^1 + B\left(1-\frac{1}{\sqrt{3}}i\right)^1 + 3$$

$$\Rightarrow 1 = A+B+\frac{1}{\sqrt{3}}i(A-B)$$

By comparing the imaginary parts,

$$0 = A-B$$

and $1 = A+B$

This gives us $A = 1/2$, $B = 1/2$ (1)

So we have

$$U_n = \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}i\right)^n + \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}i\right)^n + 3n$$

$$\Rightarrow U_n = \frac{1}{2}\left(\frac{3+\sqrt{3}i}{3}\right)^n + \frac{1}{2}\left(\frac{3-\sqrt{3}i}{3}\right)^n + 3n \quad (1)$$

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7. The set $G = \mathbb{R} - \left\{-\frac{3}{2}\right\}$ with the operation of $x \bullet y = 3(x+y+1) + 2xy$ forms a group.

(a) Determine the identity element of this group. **(2)**

(b) Determine the inverse of a general element x in this group. **(3)**

(c) Explain why the value $-\frac{3}{2}$ must be excluded from G in order for this to be a group. **(1)**

a) $x \cdot e = x$

$x = 3(x+e+1) + 2xe$ 1

$\Rightarrow x = 3x + 3e + 3 + 2xe$

$\Rightarrow -2x - 3 = e(3+2x)$

$\Rightarrow e = -1$ 1

b) $x \cdot z = e$

$x \cdot z = -1$

$-1 = 3(x+z+1) + 2xz$ 1

$\Rightarrow -3x - 4 = z(2x+3)$ 1

$\Rightarrow z = \dfrac{-3x-4}{2x+3}$

$\Rightarrow x^{-1} = \dfrac{-3x-4}{2x+3}$ 1

c) $x = -3/2$ would give a denominator of 0, which we cannot have. Hence, no inverse at $x = -3/2$. 1

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8.

$$I_n = \int_0^2 (x-2)^n e^{4x}\, dx \qquad n \geqslant 0$$

(a) Prove that for $n \geqslant 1$

$$I_n = -a^{n-2} - \frac{n}{4} I_{n-1}$$

where a is a constant to be determined.

(4)

(b) Hence determine the exact value of

$$\int_0^2 (x-2)^2 e^{4x}\, dx$$

(3)

a) We use Integration by parts.

Let $u = (x-2)^n \qquad v' = e^{4x}$

$u' = n(x-2)^{n-1} \qquad v = \frac{1}{4}e^{4x}$

$I_n = \left[\frac{1}{4}e^{4x}(x-2)^n\right]_0^2 - \int_0^2 \frac{n}{4}e^{4x}(x-2)^{n-1}\, dx$ (1)

$= \frac{1}{4}e^8(0)^n - \frac{1}{4}(-2)^n - \frac{1}{4}n\int_0^2 e^{4x}(x-2)^{n-1}\, dx$ (1)

$= -\frac{1}{4}(-2)^n - \frac{1}{4}n I_{n-1}$ (1)

$= -(-2)^{n-2} - \frac{1}{4}n I_{n-1}$ (1) so $a = -2$

b) We use part a with $n=2$ and $n=1$

$I_0 = \int_0^2 e^{4x}\, dx = \left[\frac{1}{4}e^{4x}\right]_0^2 = \frac{1}{4}e^8 - \frac{1}{4}$ (1)

$I_1 = -(-2)^{-1} - \frac{1}{4}\left(\frac{1}{4}e^8 - \frac{1}{4}\right) = \frac{9}{16} - \frac{1}{16}e^8$ (1)

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Question 8 continued

$I_2 = -(-2)^0 - \frac{2}{4}\left(\frac{9}{16} - \frac{1}{16}e^8\right)$

$= \frac{1}{32}e^8 - \frac{41}{32}$ 1

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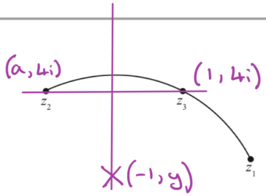

Figure 1

Figure 1 shows a locus in the complex plane.

The locus is an arc of a circle from the point represented by $z_1 = 3 + 2i$ to the point represented by $z_2 = a + 4i$, where a is a constant, $a \neq 1$

Given that

- the point $z_3 = 1 + 4i$ also lies on the locus
- the centre of the circle has real part equal to -1

(a) determine the value of a.

(2)

(b) Hence determine a complex equation for the locus, giving any angles in the equation as positive values.

(3)

a) From the diagram, see that -1 is the midpoint of a and 1. So

$$\frac{a+1}{2} = -1 \quad 1 \quad \Rightarrow \quad a = -3 \quad 1$$

b) $$\arg\left(\frac{z - z_1}{z - z_2}\right) = \theta \quad 1$$

$$\Rightarrow \arg\left(\frac{z - (3+2i)}{z - (-3+4i)}\right) = \theta$$

$$\Rightarrow \arg\left(\frac{1+4i-(3+2i)}{1+4i-(-3+4i)}\right) = \theta$$

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Question 9 continued

$$\Rightarrow \arg\left(\frac{-2+2i}{4}\right) = \theta$$

$$\Rightarrow \arg\left(-\frac{1}{2} + \frac{1}{2}i\right) = \theta$$

$$\Rightarrow \theta = \frac{3\pi}{4} \quad (1)$$

$$\arg\left(\frac{z-3-2i}{z+3-4i}\right) = \frac{3\pi}{4} \quad (1)$$

(Total for Question 9 is 5 marks)

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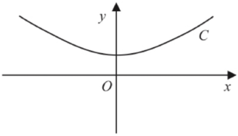

Figure 2

A solid playing piece for a board game is modelled by rotating the curve C, shown in Figure 2, through 2π radians about the x-axis.

The curve C has equation

$$y = \sqrt{1 + \frac{x^2}{9}} \qquad -4 \leqslant x \leqslant 4$$

with units as centimetres.

(a) Show that the total surface area, S cm^2, of the playing piece is given by

$$S = p\pi \int_{-4}^{4} \sqrt{81 + 10x^2}\, dx + q\pi$$

where p and q are constants to be determined.

(6)

Using the substitution $x = \frac{9}{\sqrt{10}} \sinh u$, or another algebraic integration method, and showing all your working,

(b) determine the total surface area of the playing piece, giving your answer to the nearest cm^2

(6)

a) In the formula book, we have

$$S_x = 2\pi \int y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx \quad 1$$

a) $$\frac{dy}{dx} = \frac{1}{2}\left(1 + \frac{x^2}{9}\right)^{-1/2}\left(\frac{2x}{9}\right) \quad 1$$ (by Chain Rule)

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(1 + \frac{x^2}{9}\right)^{-1}\left(\frac{x^2}{81}\right)$$

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Question 10 continued

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{81}\left(\frac{9+x^2}{9}\right)^{-1}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{81}\left(\frac{9}{9+x^2}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{9}\left(\frac{x^2}{9+x^2}\right)$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{9}\left(\frac{x^2}{9+x^2}\right)$$

$$\Rightarrow S_x = 2\pi\int_{-4}^{4}\left(1+\frac{x^2}{9}\right)^{1/2}\left(1+\frac{1}{9}\left(\frac{x^2}{9+x^2}\right)\right)^{1/2} dx \quad (1)$$

$$\Rightarrow S_x = 2\pi\int_{-4}^{4}\left[\left(1+\frac{x^2}{9}\right)\left(1+\frac{1}{9}\left(\frac{x^2}{9+x^2}\right)\right)\right]^{1/2} dx$$

$$\Rightarrow S_x = 2\pi\int_{-4}^{4}\left[\left(\frac{9+x^2}{9}\right)\frac{1}{9}\left(9+\frac{x^2}{9+x^2}\right)\right]^{1/2} dx$$

$$\Rightarrow S_x = 2\pi\int_{-4}^{4}\frac{1}{3}\left[\left(\frac{9+x^2}{9}\right)\left(\frac{10x^2+81}{9+x^2}\right)\right]^{1/2} dx$$

$$\Rightarrow S_x = \frac{2\pi}{3}\int_{-4}^{4}\frac{1}{3}\left(10x^2+81\right)^{1/2} dx \quad (1)$$

$$\Rightarrow S_x = \frac{2\pi}{9}\int_{-4}^{4}\sqrt{10x^2+81}\, dx$$

We are asked for the total surface area, so have to add the area of the two circles on either end.

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Question 10 continued

Each circle has radius

$\sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

So we have $2 \times \left(\pi \times \left(\frac{5}{3}\right)^2\right) = \frac{50}{9}\pi$ (1)

So $S = \frac{2\pi}{9}\int_{-4}^{4} \sqrt{10x^2+81}\,dx + \frac{50\pi}{9}$ (1)

b) $x = \frac{9}{\sqrt{10}}\sinh u$

$\Rightarrow \frac{dx}{du} = \frac{9}{\sqrt{10}}\cosh u$ (1)

$\Rightarrow dx = \frac{9}{\sqrt{10}}\cosh u\,du$

Finding the limits:

$4 = \frac{9}{\sqrt{10}}\sinh u \Rightarrow u = 1.141$

$-4 = \frac{9}{\sqrt{10}}\sinh u \Rightarrow u = -1.141$ (1)

Also, $x^2 = \frac{81}{10}\sinh^2 u$, so our integral becomes

$S = \frac{2\pi}{9}\int_{-1.141}^{1.141} \sqrt{81\sinh^2 u + 81} \cdot \frac{9}{\sqrt{10}}\cosh u\,du + \frac{50\pi}{9}$ (1)

$= \frac{2\pi}{9}\int_{-1.141}^{1.141} 9\sqrt{\sinh^2 u + 1} \cdot \frac{9}{\sqrt{10}}\cosh u\,du + \frac{50\pi}{9}$

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Question 10 continued

Recall that $\sinh^2 u + 1 = \cosh^2 u$

$\Rightarrow S = \frac{18}{\sqrt{10}}\pi \int_{-1.144}^{1.144} \cosh^2 u \, du + \frac{50\pi}{9}$

Recall that $\cosh^2 u = \frac{1}{2}\cosh 2u + \frac{1}{2}$

$\Rightarrow S = \frac{9}{\sqrt{10}}\pi \int_{-1.141}^{1.144} \cosh 2u + 1 \, du$ (1)

$\Rightarrow S = \frac{9}{\sqrt{10}}\pi \left[\frac{1}{2}\sinh 2u + u\right]_{-1.141}^{1.141} + \frac{50\pi}{9}$ (1)

$\Rightarrow S = 81 cm^2$ to the nearest cm. (1)

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